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LETTER TO THE EDITOR

On the algebraic structures connected with the linear Poisson brackets of hydrodynamics type

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Abstract. The generalized form of the Kac formula for Verma modules associated with linear brackets of hydrodynamics type is proposed. Second cohomology groups of the generalized Virasoro algebras are calculated. Connection of the central extensions with the problem of quantization of hydrodynamics brackets is demonstrated.

Poisson brackets of hydrodynamics type (PBHT)

$$\{u^i(x), u^j(y)\} = g^{ij}(u(x))\delta'(x-y) + u^k_x b^{ij}_k(u(x))\delta(x-y) \tag{1}$$

(here and below we assumed a summation on repeat indexes) were introduced and studied in [1, 2] to construct a theory of conservative systems of hydrodynamics type and to develop a Bogolubov–Whitham method of averaging Hamiltonian field-theoretic systems. We refer to the recent expository article [3] and the extensive bibliography therein. In [4] Novikov and the first author considered gave a classification of these Poisson brackets depending linearly on the fields u^j relative to linear change $u^k = A^k_j w^j$. Some examples were discussed in [5, 6].

For the reader's convenience we recall some construction from [4]. The simplest local Lie algebras arising from the brackets of hydrodynamics type are especially interesting, where, according to [4], in the case when all metrics are linear in u we have

$$\begin{aligned} &g^{ij} = C^{ij}_k u^k + g^{ij}_0 \\ &b^{ij}_k = \text{const} \qquad g^{ij}_0 = \text{const} \\ &C^{ij}_k = b^{ij}_k + b^{ji}_k. \end{aligned} \tag{2}$$

The linear (homogeneous) part of such PBHT determines some very interesting classes of the infinite-dimensional Lie algebras ('hydrodynamic algebras'): for two vector-functions $f(x)$ and $g(x)$ with N components f_p, g_q we may define the commutator in the 'local translation-invariant first-order Lie algebra' or the hydrodynamic algebra

$$[f, g]_k(x) = b^{pq}_k(f'_p(x)g_q(x) - g'_p(x)f_q(x)). \tag{3}$$

A bracket (1) or Lie algebra (3), linear in the fields, is called *symmetric* if $b^{ij}_k = b^{ji}_k$. Here f_p are adjoint variables for u^i.

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It is useful to introduce a new algebra $\boldsymbol{B}$ as a mutliplication in N-space M with basis $e^1, e^2, \ldots, e^N$

$$e^i e^j = b^{ij}_k e^k. \tag{4}$$

For the functions $f(x) = f_p(x)e^p$ and $g(x) = g_q(x)e^q$ we write (3) in the form $f'g - g'f$ using multiplication (4) in the algebra $\boldsymbol{B}$. The tensor b^{ij}_k defines, by (3), a local translationally invariant Lie algebra of first order if, and only if, the multiplicaton law (4) defines a finite-dimensional algebra $\boldsymbol{B}$ in which the following indentities hold:

$$a, b, c \in B \qquad (ab)c = (ac)b \qquad (ab)c - a(bc) = (ba)c - b(ac). \tag{5}$$

In the symmetric case $2b^{ij}_k = 2b^{ji}_k = C^{ij}_k$ this algebra is commutative and associative.

Remark. After introducing an operation $[a, b] = ab - ba$ on $\boldsymbol{B}$ it is Lie algebra and as was first proved in [7] this is a solvable Lie algebra. If a finite-dimensional algebra $\boldsymbol{B}$ (4), (5) is commutative then it is automatically associative, and if it has the right unit then it is commutative. The theory of extensions for the algebras $\boldsymbol{B}$ (4), (5) was constructed in [4, 8]. The method of construction of the wide class of such algebras was proposed by Gelfand: if algebra A is commutative and associative, ∂-differentiation of A then multiplication $a \circ b = a(\partial b)$ satisfied (5).

The formula (1) defines Poisson bracket if, and only if, an algebra (4) satisfies (5) and for the following symmetric bilinear product

$$(e^i, e^j)_0 = g^{ij}_0$$

we have

$$(ab, c)_0 = (a, cb)_0 \tag{6}$$

for all a, b, $c \in \boldsymbol{B}$. In this case the 'quasifrobenius property' (6) holds true for all symmetric belinear products with matrix $g^{ij}(u) = C^{ij}_k u^k + g^{ij}_0$ for any u^i. If the algebra $\boldsymbol{B}$ is commutative and has the unit then we have the classical *Frobenius algebra.*

Poisson bracket (1) is called *non-degenerate* if the pseudo-Riemannian metric $g^{ij} = C^{ij}_k u^k + g^{ij}_0$ is non-degenerate at a 'generic point':

$$\det(g^{ij}) := P_N(u^1, u^2, \ldots, u^N) \neq 0.$$

Lie algebra (3) and a finite-dimensional algebra $\boldsymbol{B}$ (5) are called *non-degenerate* if the pseudo-Riemannian metric $g^{ij} = C^{ij}_k u^k$ is non-degenerate at a '*generic point*', where $b^{ij}_k + b^{ji}_k = C^{ij}_k$. From the main theorem of [1] we know that in a non-degenerate case these pseudo-Riemannian metrics have a vanishing curvature.

For the vector-valued functions periodic in x, by passing to an expansion in Fourier series, we obtain a basis (L^i_n) for the algebra (3) with the relations

$$[L^i_n, L^j_m] = (nb^{ij}_k - mb^{ji}_k) L^k_{m+n} \tag{7}$$

which we call the generalized Witt algebra. For this algebra we have the following generalization of the Gelfand–Fuks theorem [9] on the central extension of the algebra of vector fields on the circle:

Theorem 1. If the algebra (4) is commutative and has the unit then

$$H^2(V) = \boldsymbol{B}^*$$

where V-algebra (7) and H^2-second cohomology group of Lie algebra and $\boldsymbol{B}^*$ is the dual space for the algebra (4). The all central extensions of (7) have the following form

$$[L_n^i, L_m^j]' = (n-m)b_k^{ij}L_{m+n}^k + b_k^{ij}l^k \frac{(n^3-n)}{12}\delta_{n+m,0}Z \tag{8}$$

where Z-central element and $l=(l_i)\in \boldsymbol{B}^*$.

In the case when $\boldsymbol{B}=\boldsymbol{C}$ (algebra of complex numbers) we have Gelfand–Fuks theorem.

For the algebra (8) we may consider Verma module $V_{h,l}$, $h, l\in\boldsymbol{B}^*$, over this algebra: $V_{h,l}$ free generate by $|v\rangle$ over L_n^i with $n>0$, $i=1,\ldots,N$ and

$$L_n^i|v\rangle = 0 \qquad n<0$$
$$L_0^i|v\rangle = h(e^i)|v\rangle$$
$$Z|v\rangle = |v\rangle.$$

An element of the Verma module is singular if it generates the Verma submodule, i.e. it is an eigenvector for all L_0^i, and annihilated by all L_n^i, $n<0$.

Theorem 2. For the algebra (8) with the unitial $\boldsymbol{B}$ Verma module is reducible if, and only if, it has the singular vector.

When algebra (4) is commutative and has the unit then for (8) from the root decomposition of $\boldsymbol{B}$ we have the following analogue of the famous Kac–Feigin–Fuks criteria:

Theorem 3. Verma module $V_{h,l}$ is reducible if, and only if, in algebra $\boldsymbol{B}$ (4) exist one-dimensional ideal $\langle a\rangle$ such that if $a^2=0$ then $h(a)=l(a)(\alpha^2-1)/2$ for some $\alpha\in N_+$, or if $aa=\mu a$ with $\mu\in\boldsymbol{C}$, $\mu\neq 0$ then $\bar{h}:=h(a)/\mu$, $\bar{c}:=l(a)/\mu$ satisfied to Kac condition [10] for the ordinary Virasoro algebra.

Now we consider the problem of quantization of the PBHT (1). Its decision may be found by means of the change of variables $u=u(v)$ such that in the new variables v bracket (1) is constant. After canonical quantizations of the constant bracket we can come back to the old variables u. But we run up against the problem of ordering. Thus we need the simplest possible change of the variables. The linear change is not suitable. Since according to [1] in the case of non-degenerate algebras (3), (4) the metric $g^{ij}=C_k^{ij}u^k$ must have zero curvature, we appeal to changes $u(v)$, which are now nonlinear, where metric in the new coordinates $(v^1,\ldots,v^N)$ is constant

$$g^{ij}(u(v)) = g_0^{\alpha\beta}(\partial u^i/\partial v^\alpha)(\partial u^j/\partial v^\beta) \qquad g_0^{\alpha\beta}=\text{constant}.$$

We consider the purely quadratic changes

$$u^i = \tfrac{1}{2}F_{\alpha\beta}^i v^\alpha v^\beta. \tag{9}$$

Then for a change (9) to reduce the non-degenerate metric of zero curvature $g^{ij}=C_k^{ij}u^k$ (from PBHT) to constant form it is necessary and sufficient that the following conditions hold: $b_k^{ij}=b_k^{ji}$; F and g_0^{ij} determine a Frobenius representation of the algebra (4), where the $F_{\alpha\gamma}^i$ give a representation of the basis e^i of the algebra in the form of linear operator in v-space which are selfadjoint in this inner product, so that

$$e^i\to(F^i)_\beta^\alpha = g_\gamma^\alpha F_{\gamma\beta}^i \qquad F^iF^j = C_k^{ij}F^k/2 \qquad \det(F_{\alpha\beta}^i v^\beta \neq 0).$$

Thus if the algebra $\boldsymbol{B}$ (4) is commutative and non-degenerate (in this case it has the unit) by the quadratic changes we obtained constant bracket. After the linear change of variables from v to $(\phi)'$ we have

$$\{(\phi^i(x))', (\phi^j(y))'\} = \delta^{ij}\delta'(x-y).$$

Dirac quantization leads to the theory of free fields in 2D quantum field theory, and the Fourier components of $(\phi)'$ form the following famous algebra ($i = 1, \ldots, N, s \in \boldsymbol{Z}$):

$$[a^i(s), a^j(k)] = k\delta^{ij}\delta_{s+k,0}.$$

After choosing the ordering procedure for $a^i(s)$ we obtain for (9) not algebra (7), but the algebra (8), and the ordering procedures are in one to one correspondence with central extensions of (7). Thus algebra (8) is the quantization of PBHT (1) in the case when algebra (4) is symmetric and non-degenerate.

Remark. It will be very interesting to investigate the algebras (4) and PBHT with another type of change of variables $u = u(v)$ such that in the new variables v bracket (1) is constant. Some examples of such non-commutative algebras (for the Poisson brackets of one-dimensional hydrodynamics) were investigated in [4].

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